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The Schur complements of γ-diagonally and product γ-diagonally dominant matrix and their disc separation[☆]

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ABSTRACT

In this paper, we obtain some estimates for the γ-diagonally and product γ-diagonally dominant degree of the Schur complement of matrices. As application we present some bounds for the eigenvalues of Schur complement by the entries of the original matrix. Based on these results, we give a kind of iteration called the Schur-based iteration, which can solve large scale linear systems though reducing the order by the Schur complement and can compute out the results faster.

1. Introduction and notations

To begin with, we first recall some notations and definitions. Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices, $N = \{1, 2, \ldots, n\}$ and $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, where $n \geqslant 2$. Denote $|A| = (|a_{ij}|)$ and

$$P_i(A) = \sum_{j\in N,\, j\neq i} |a_{ij}|, \quad S_i(A) = \sum_{j\in N,\, j\neq i} |a_{ji}|, \quad i = 1, 2, \ldots, n.$$

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Take

$$N_r(A) = \{i|i \in N,\ |a_{ii}| > P_i(A)\};\ N_c(A) = \{j|j \in N,\ |a_{jj}| > S_j(A)\}.$$

The comparison matrix of A, denoted by $\mu(A) = (t_{ij})$, is defined to be

$$t_{ij} = \begin{cases} |a_{ij}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}$$

Recall that A is a (row) diagonally dominant matrix (D_n) if for all $i = 1, 2, \ldots, n$,

$$|a_{ii}| \geqslant P_i(A). \tag{1.1}$$

A is a doubly diagonally dominant matrix (DD_n) if for all $i, j \in N,\ i \neq j$,

$$|a_{ii}||a_{jj}| \geqslant P_i(A)P_j(A). \tag{1.2}$$

A is a γ-diagonally dominant matrix $\left(D_n^\gamma\right)$ if there exists $\gamma \in [0, 1]$ such that

$$|a_{ii}| \geqslant \gamma P_i(A) + (1 - \gamma)S_i(A), \quad \forall i \in N. \tag{1.3}$$

And A is called a product γ-diagonally dominant matrix $\left(PD_n^\gamma\right)$ if there exists $\gamma \in [0, 1]$ such that

$$|a_{ii}| \geqslant [P_i(A)]^\gamma [S_i(A)]^{1-\gamma}, \quad \forall i \in N. \tag{1.4}$$

If all inequalities in (1.1)–(1.4) hold, A is said to be strictly (row) diagonally dominant (SD_n), strictly doubly diagonally dominant (SDD_n), strictly γ-diagonally dominant $\left(SD_n^\gamma\right)$ and strictly product γ-diagonally dominant $\left(SPD_n^\gamma\right)$, respectively. For $1 \leqslant i \leqslant n$ and $\gamma \in [0, 1]$, we call $|a_{ii}| - P_i(A)$, $|a_{ii}| - \gamma P_i(A) - (1 - \gamma)S_i(A)$ and $|a_{ii}| - [P_i(A)]^\gamma [S_i(A)]^{1-\gamma}$ the ith (row) dominant degree, γ-dominant degree and product γ-dominant degree of A, respectively.

For nonempty index sets $\alpha, \beta \subseteq N$ whose elements are both conventionally arranged in increasing order, we denote by $|\alpha|$ the cardinality of α and $\alpha' = N - \alpha$ the complement of α in N. We write $A(\alpha, \beta)$ to mean the submatrix of $A \in \mathbb{C}^{n\times n}$ lying in the rows indexed by α and the columns indexed by β. $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Assuming that $A(\alpha)$ is nonsingular, the Schur complement of A with respect to $A(\alpha)$, which is denoted by $A/A(\alpha)$ or simply A/α, is defined to be

$$A/\alpha = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha'). \tag{1.5}$$

It is known that the Schur complements of positive semidefinite matrices are positive semidefinite; the same is true of M-matrix, H-matrix and the inverse of M-matrix (see, e.g., [1]). Carlson and Markham showed that the Schur complements of strictly diagonally dominant matrices are strictly diagonally dominant (see [2]). Li, Tsatomeros and Ikramov independently proved that the Schur complements of doubly diagonally dominant matrices are doubly diagonally dominant (see, e.g., [3,4]). Liu and Huang obtained that the Schur complements of generalized diagonally dominant matrices are generalized diagonally dominant (see [5]). Liu, Huang, Zhang, Zhu and Smith got some upper and lower bounds for eigenvalues, singular values and determinants of Schur complement (see, e.g., [6–10]). These very properties have been repeatedly used for the convergence of iterations in numerical analysis and for deriving matrix inequalities in matrix analysis (see, e.g., [11, p. 508], [12, p. 58] or [8]).

Meanwhile, investigating the distribution for the eigenvalues of the Schur complement is of great significance. For example, consider a non-homogeneous system of linear equation $Mx = b$ with a nonsingular leading principal submatrix. Partition M as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is supposed to be nonsingular. Partition $x = \left(x_1^T x_2^T\right)^T$ and $b = \left(b_1^T b_2^T\right)^T$ conformably with M. Then the linear system $Mx = b$ is equivalent to the pair of linear systems

$$Ax_1 + Bx_2 = b_1, \tag{1.6}$$

$$Cx_1 + Dx_2 = b_2. \tag{1.7}$$

If we multiply (1.6) by $-CA^{-1}$ and add it to (1.7), the vector variable x_1 is eliminated and we obtain a linear system of smaller size:

$$(D - CA^{-1}B)x_2 = b_2 - CA^{-1}b_1. \tag{1.8}$$

We construct an iterative equation as the following:

$$x_2^{k+1} = [I - (D - CA^{-1}B)]x_2^k + b_2 - CA^{-1}b_1, \quad k = 1, 2, \ldots \tag{1.9}$$

It is known that Eq. (1.9) is convergent if and only if the spectral radius $\rho(I - (D - CA^{-1}B)) < 1$. So, if $\rho(D - I - CA^{-1}B) < 1$, (1.9) is convergent and we can solve (1.8) by iteration, thus the original system. Therefore, if the eigenvalues of the Schur complement $(D - I - CA^{-1}B)$ can be estimated by the entries of the original matrix, it is easy to know whether a linear system could be transformed into a smaller one which can be solved by iteration. We call this kind of iteration the Schur-based iteration which converts the original system into two smaller ones by the Schur complement. The advantages of this kind of Schur-based iteration will be shown in this paper.

In this paper, we obtain some estimates for the γ-diagonally and product γ-diagonally dominant degree of the Schur complements of matrices. As application, we give some bounds for the eigenvalues of the Schur complement by the entries of the original matrix instead of those of the Schur complement. Particularly, we obtain that the eigenvalues of the Schur complements are located in the *Gerschgorin Circles* of the original matrices under certain conditions. Further more, we give an example to show the advantages of the Schur-based iteration.

2. The γ-diagonally and product γ-diagonally dominant degree for Schur complement

As is known, the Schur complements of diagonally dominant matrices and doubly diagonally dominant matrices are diagonally dominant and doubly diagonally dominant, respectively (see, e.g, [2,3]). But for the γ-diagonally or product γ-diagonally dominant matrices, these corresponding properties are not always true (see, e.g. [14]), which is shown by the following example.

Example 1. Let

$$A = \begin{pmatrix} 10 & -1 & 1 \\ 0.1 & 2 & 2.1 \\ 16 & 0.5 & 20 \end{pmatrix}, \quad \alpha = \{1\}.$$

Then

$$\begin{aligned}
&|a_{11}| = 10, \quad P_1(A) = 2, \quad Q_1(A) = 16.1; \\
&|a_{22}| = 2, \quad P_2(A) = 2.2, \quad Q_2(A) = 1.5; \\
&|a_{33}| = 20, \quad P_3(A) = 16.5, \quad Q_3(A) = 3.1; \\
&A/\alpha = \begin{pmatrix} 2.01 & 2.09 \\ 2.1 & 18.4 \end{pmatrix}.
\end{aligned}$$

Choose $\gamma_0 = 0.5$. It is obvious that $A \in SD_3^{\gamma_0}$ and $A \in SPD_3^{\gamma_0}$. Since $P_1(A/A(\alpha)) = 2.09 > 2.01$ and $Q_1(A/A(\alpha)) = 2.1 > 2.01$, there exists no $\gamma \in [0, 1]$ satisfying $A/A(\alpha) \in SD_2^{\gamma}$. Hence $A/A(\alpha)$ is not γ-diagonally dominant. Similarly, since for any $\gamma \in [0, 1]$, $2.09^{\gamma}2.1^{1-\gamma} > 2.01^{\gamma}2.01^{1-\gamma} = 2.01$, $A/A(\alpha)$ is not product γ-diagonally dominant.

In this section, we obtain some estimates for the γ-diagonally and product γ-diagonally dominant degree of the Schur complement under some conditions. And we give some conditions under which the Schur complement of the γ-diagonally and product γ-diagonally dominant matrices must be γ-diagonally dominant and product γ-diagonally, respectively.

For this, we first recall the following results.

Lemma 1 (see [13, p. 117, 131]). *If A is a H-matrix, then* $[\mu(A)]^{-1} \geqslant |A^{-1}|$.

Lemma 2 (see [13, p. 114]). *If $A \in SD_n$ or SDD_n. Then $\mu(A) \in \mathbb{M}_n$, i.e., $A \in \mathbb{H}_n$, where $\mathbb{M}_n$ and $\mathbb{H}_n$ denote the sets of M-matrices and H-matrices, respectively.*

Note that for a number $a \in \mathbb{C}$, a nonsingular matrix $S \in \mathbb{C}^{(n-1)\times(n-1)}$ and $x, y \in \mathbb{C}^{n-1}$,

$$\det\left(\begin{bmatrix} a & x^T \\ y & S \end{bmatrix}\right) = \frac{1}{\det(S)}\det(a - x^T S^{-1} y).$$

We have the following lemma.

Lemma 3. *Let $A \in \mathbb{C}^{n\times n}, \alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N, \alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}, k = |\alpha| < n, l = n - k$. Denote $A/\alpha = (a'_{ts})$.*

(i) *If $\alpha \subseteq N_r(A)$, then for all $1 \leqslant t \leqslant l$,*

$$\left|(a_{j_t i_1}, \ldots, a_{j_t i_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| + \sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_t j_s} - (a_{j_t i_1}, \ldots, a_{j_t i_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right|$$
$$\leqslant P_{j_t}(A) - w_{j_t}. \tag{2.1}$$

(ii) *If $\alpha \subseteq N_c(A)$, then for all $1 \leqslant t \leqslant l$,*

$$\left|(a_{j_t i_1}, \ldots, a_{j_t i_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| + \sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_s j_t} - (a_{j_s i_1}, \ldots, a_{j_s i_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right|$$
$$\leqslant S_{j_t}(A) - w^T_{j_t}. \tag{2.2}$$

Here we denote

$$w_{j_t} = \min_{1 \leqslant v \leqslant k}\frac{|a_{i_v i_v}| - P_{i_v}(A)}{|a_{i_v i_v}|}\sum_{u=1}^{k}|a_{j_t i_u}|; \quad w^T_{j_t} = \min_{1 \leqslant v \leqslant k}\frac{|a_{i_v i_v}| - S_{i_v}(A)}{|a_{i_v i_v}|}\sum_{u=1}^{k}|a_{i_u j_t}|.$$

Proof. Since $\alpha \subseteq N_r(A), A(\alpha) \in SD_k$. By Lemmas 1 and 2,

$$\{\mu[A(\alpha)]\}^{-1} \geqslant \left|[A(\alpha)]^{-1}\right|.$$

In a similar way with the proof of Theorem 1 in [7], we give the proof of (2.1) as the following

$$\left|(a_{j_t i_1}, \ldots, a_{j_t i_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t}\end{pmatrix}\right| + \sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_t j_s} - (a_{j_t i_1}, \ldots, a_{j_t i_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right|$$
$$\leqslant \sum_{\substack{s=1\\ s\neq t}}^{l}|a_{j_t j_s}| + \sum_{s=1}^{l}\left|(a_{j_t i_1}, \ldots, a_{j_t i_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s}\end{pmatrix}\right|$$
$$\leqslant \sum_{\substack{s=1\\ s\neq t}}^{l}|a_{j_t j_s}| + \sum_{s=1}^{l}(|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|)\{\mu[A(\alpha)]\}^{-1}\begin{pmatrix} |a_{i_1 j_s}| \\ \vdots \\ |a_{i_k j_s}|\end{pmatrix}$$

$$= P_{j_t}(A) - w_{j_t} - \left[\sum_{u=1}^{k} |a_{j_t i_u}| - w_{j_t} - (|a_{j_t i_1}|, \ldots, |a_{j_t i_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} \sum_{s=1}^{l} |a_{i_1 j_s}| \\ \vdots \\ \sum_{s=1}^{l} |a_{i_k j_s}| \end{pmatrix} \right]$$

$$= P_{j_t}(A) - w_{j_t} - \frac{1}{\det\{\mu[A(\alpha)]\}} \det \begin{pmatrix} \sum_{u=1}^{k} |a_{j_t i_u}| - w_{j_t} & -|a_{j_t i_1}| & \cdots & -|a_{j_t i_k}| \\ -\sum_{s=1}^{l} |a_{i_1 j_s}| & & & \\ \vdots & & \mu[A(\alpha)] & \\ -\sum_{s=1}^{l} |a_{i_k j_s}| & & & \end{pmatrix}$$

$$= P_{j_t}(A) - w_{j_t} - \frac{1}{\det\{\mu[A(\alpha)]\}} \det B_1.$$

Since $\alpha \subseteq N_r(A)$, from Lemma 4 in [7] we obtain $B_1 \in SDD_{k+1}$ and $\det B_1 \geqslant 0$. Thus we obtain (2.1).

With a similar proof we obtain (2.2). □

Lemma 4. *Let $a > b, c > b, b > 0$ and $0 \leqslant r \leqslant 1$. Then*

$$a^r c^{1-r} \geqslant (a-b)^r (c-b)^{1-r} + b. \tag{2.3}$$

Proof. Let $s = a - b, t = c - b$, by the *Hölder* inequality we have

$$a^r c^{1-r} = (s+b)^r (t+b)^{1-r} \geqslant s^r t^{1-r} + b^r b^{1-r} = (a-b)^r (c-b)^{1-r} + b. \quad \square$$

Now we are ready to give the main results in this section.

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$, $N_r(A) \cap N_c(A) \neq \emptyset$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \cap N_c(A)$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$, $k = |\alpha| < n$, $l = n - k$ and $A/\alpha = (a'_{ts})$. Then for all $1 \leqslant t \leqslant l$, $0 \leqslant \gamma \leqslant 1$,*

$$\begin{aligned} |a'_{tt}| - \gamma P_t(A/\alpha) - (1-\gamma) S_t(A/\alpha) &\geqslant |a_{j_t j_t}| - \gamma P_{j_t}(A) - (1-\gamma) S_{j_t}(A) \\ &\quad + \gamma w_{j_t} + (1-\gamma) w_{j_t}^T \\ &\geqslant |a_{j_t j_t}| - \gamma P_{j_t}(A) - (1-\gamma) S_{j_t}(A) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} |a'_{tt}| + \gamma P_t(A/\alpha) + (1-\gamma) S_t(A/\alpha) &\leqslant |a_{j_t j_t}| + \gamma P_{j_t}(A) + (1-\gamma) S_{j_t}(A) - \gamma w_{j_t} - (1-\gamma) w_{j_t}^T \\ &\leqslant |a_{j_t j_t}| + \gamma P_{j_t}(A) + (1-\gamma) S_{j_t}(A). \end{aligned} \tag{2.5}$$

Proof. Since $\alpha \subseteq N_r(A) \cap N_c(A)$, by the definition of the Schur complement,

$$\begin{aligned} &|a'_{tt}| - \gamma P_t(A/\alpha) - (1-\gamma) S_t(A/\alpha) \\ &= |a'_{tt}| - \gamma \sum_{\substack{s=1 \\ s\neq t}}^{l} |a'_{ts}| - (1-\gamma) \sum_{\substack{s=1 \\ s\neq t}}^{l} |a'_{st}| \\ &= \left| a_{j_t j_t} - (a_{j_t i_1}, \ldots, a_{j_t i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| - \gamma \sum_{\substack{s=1 \\ s\neq t}}^{l} \left| a_{j_t j_s} - (a_{j_t i_1}, \ldots, a_{j_t i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right| \\ &\quad - (1-\gamma) \sum_{\substack{s=1 \\ s\neq t}}^{l} \left| a_{j_s j_t} - (a_{j_s i_1}, \ldots, a_{j_s i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| \end{aligned}$$

$$
\begin{aligned}
&\geqslant |a_{j_tj_t}| - \left|(a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right| - \gamma\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_tj_s} - (a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_s}\\ \vdots\\ a_{i_kj_s}\end{pmatrix}\right|\\
&\quad - (1-\gamma)\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_sj_t} - (a_{j_si_1},\ldots,a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|\\
&= |a_{j_tj_t}| - \gamma\left[\left|(a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|\right.\\
&\quad \left.+\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_tj_s} - (a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_s}\\ \vdots\\ a_{i_kj_s}\end{pmatrix}\right|\right]\\
&\quad - (1-\gamma)\left[\left|(a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|\right.\\
&\quad \left.+\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_sj_t} - (a_{j_si_1},\ldots,a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|\right]\\
&\geqslant |a_{j_tj_t}| - \gamma\left(P_{j_t}(A) - w_{j_t}\right) - (1-\gamma)\left(S_{j_t}(A) - w_{j_t}^T\right) \quad \text{(by Lemma 3)}\\
&= |a_{j_tj_t}| - \gamma P_{j_t}(A) - (1-\gamma)S_{j_t}(A) + \gamma w_{j_t} + (1-\gamma)w_{j_t}^T.
\end{aligned}
$$

Thus we get (2.4).

With a similar proof we get (2.5). □

By Theorem 1 we have the following corollary.

Corollary 1. *Let $A \in SD_n^{\gamma}$ and $N_r(A) \cap N_c(A) \neq \emptyset$. Then for any $\alpha \subseteq N_r(A) \cap N_c(A)$ satisfying $|\alpha| < n$, $A/\alpha \in SD_{n-|\alpha|}^{\gamma}$.*

Proof. By (2.4), when $A \in SD_n^{\gamma}$,

$$|a'_{tt}| - \gamma P_t(A/\alpha) - (1-\gamma)S_t(A/\alpha) \geqslant |a_{j_tj_t}| - \gamma P_{j_t}(A) - (1-\gamma)S_{j_t}(A) > 0.$$

Thus we get the result. □

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$, $N_r(A) \cap N_c(A) \neq \emptyset$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \cap N_c(A)$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$, $k = |\alpha| < n$, $l = n - k$ and $A/\alpha = (a'_{ts})$. Then for all $1 \leqslant t \leqslant l$, $0 \leqslant \gamma \leqslant 1$,*

$$
\begin{aligned}
|a'_{tt}| - P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha) &\geqslant |a_{j_tj_t}| - (P_{j_t}(A) - w_{j_t})^{\gamma}(S_{j_t}(A) - w_{j_t}^T)^{1-\gamma}\\
&\geqslant |a_{j_tj_t}| - P_{j_t}^{\gamma}(A)S_{j_t}^{1-\gamma}(A) \qquad (2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
x|a'_{tt}| + P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha) &\leqslant |a_{j_tj_t}| + (P_{j_t}(A) - w_{j_t})^{\gamma}(S_{j_t}(A) - w_{j_t}^T)^{1-\gamma}\\
&\leqslant |a_{j_tj_t}| + P_{j_t}^{\gamma}(A)S_{j_t}^{1-\gamma}(A). \qquad (2.7)
\end{aligned}
$$

Proof. By the definition of the Schur complement,

$$
\begin{aligned}
&|a'_{tt}| - P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha)\\
&= |a'_{tt}| - \left(\sum_{\substack{s=1\\ s\neq t}}^{l} |a'_{ts}|\right)^{\gamma}\left(\sum_{\substack{s=1\\ s\neq t}}^{l} |a'_{st}|\right)^{1-\gamma}\\
&= \left| a_{j_tj_t} - (a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\\
&\quad - \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left| a_{j_tj_s} - (a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_s}\\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\gamma}\\
&\quad \cdot \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left| a_{j_sj_t} - (a_{j_si_1},\ldots,a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\right]^{(1-\gamma)}\\
&\geqslant |a_{j_tj_t}| - \left|(a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\\
&\quad - \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left| a_{j_tj_s} - (a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_s}\\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\gamma}\\
&\quad \cdot \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left| a_{j_sj_t} - (a_{j_si_1},\ldots,a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\right]^{(1-\gamma)}. \qquad (2.8)
\end{aligned}
$$

Denote

$$
h = \left|(a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|.
$$

From (2.1), (2.2) and (2.8), by Lemma 4 we have

$$
\begin{aligned}
&|a'_{tt}| - P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha)\\
&\geqslant |a_{j_tj_t}| - h - (P_{j_t}(A) - w_{j_t} - h)^{\gamma}(S_{j_t}(A) - w_{j_t}^T - h)^{1-\gamma}\\
&\geqslant |a_{j_tj_t}| - h - \left[(P_{j_t}(A) - w_{j_t})^{\gamma}(S_{j_t}(A) - w_{j_t}^T)^{1-\gamma} - h\right]\\
&= |a_{j_tj_t}| - (P_{j_t}(A) - w_{j_t})^{\gamma}(S_{j_t}(A) - w_{j_t}^T)^{1-\gamma}.
\end{aligned}
$$

Thus we get (2.6).

With a similar proof we get (2.7). □

Similar to Corollary 1, we have the following corollary by Theorem 2.

Corollary 2. *Let* $A \in SPD_n^{\gamma}$ *and* $N_r(A) \cap N_c(A) \neq \emptyset$. *Then for any* $\alpha \subseteq N_r(A) \cap N_c(A)$ *satisfying* $|\alpha| < n$, $A/\alpha \in SPD_{n-|\alpha|}^{\gamma}$.

3. Distribution for eigenvalues

In this section, we present some locations for eigenvalues of the Schur complement by the entries of the original matrix instead of those of the Schur complement.

Firstly, we give some distributions for eigenvalues of the Schur complement based on the results of [7].

Theorem 3. *Let* $A \in \mathbb{C}^{n\times n}, N_r \neq \emptyset, \alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N_r$ *and* $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$. *Then for every eigenvalue* λ *of* A/α, *there exists* $1 \leqslant t \leqslant l$ *such that*

$$|\lambda - a_{j_tj_t}| \leqslant P_{j_t}(A) - w_{j_t} \leqslant P_{j_t}(A). \tag{3.1}$$

Proof. Set $A/\alpha = (a'_{st})$. By the famous *Gerschgorin Circle Theorem*, there exists $1 \leqslant t \leqslant l$ such that

$$|\lambda - a'_{tt}| \leqslant P_t(A/\alpha).$$

Thus

$$\begin{aligned}
0 &\geqslant |\lambda - a'_{tt}| - P_t(A/\alpha) \\
&= \left|\lambda - a_{j_tj_t} + (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\
&\quad - \sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_tj_s} - (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right| \\
&\geqslant |\lambda - a_{j_tj_t}| - \left|(a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t} \\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\
&\quad - \sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_tj_s} - (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_s} \\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right| \\
&\geqslant |\lambda - a_{j_tj_t}| - \left(P_{j_t}(A) - w_{j_t}\right),
\end{aligned}$$

i.e.,

$$|\lambda - a_{j_tj_t}| \leqslant P_{j_t}(A) - w_{j_t} \leqslant P_{j_t}(A).$$

This theorem is based on the condition that $\alpha \subseteq N_r$. However, sometimes α satisfies not only $\alpha \subseteq N_r$ but also $\alpha \subseteq N_r(A) \cap N_c(A)$. As for the latter, we give some other estimates for the eigenvalues of Schur complement based on the results of Section 2. For this, we first recall the famous *OstrowskiTheorem*. □

Lemma 5 (see [15]). *Let* $A \in \mathbb{C}^{n\times n}$ *and* $0 \leqslant \gamma \leqslant 1$. *Then for every eigenvalue* λ *of* A, *there exists* $1 \leqslant i \leqslant n$ *such that*

$$|\lambda - a_{ii}| \leqslant P_i^{\gamma}(A)S_i^{1-\gamma}(A). \tag{3.2}$$

Theorem 4. *Let $A \in \mathbb{C}^{n\times n}$, $N_r(A) \cap N_c(A) \neq \emptyset$, $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N_r(A) \cap N_c(A)$, $\alpha' = N - \alpha = \{j_1, j_2, \ldots, j_l\}$ and $k = |\alpha| < n, l = n - k$. Then for any $0 \leqslant \gamma \leqslant 1$ and every eigenvalue λ of A/α, there exists $1 \leqslant t \leqslant l$ such that*

$$|\lambda - a_{j_tj_t}| \leqslant (P_{j_t}(A) - w_{j_t})^{\gamma}(S_{j_t}(A) - w_{j_t}^T)^{1-\gamma} \leqslant P_{j_t}^{\gamma}(A)S_{j_t}^{1-\gamma}(A). \tag{3.3}$$

Proof. Denote $A/\alpha = (a'_{ts})$. From Lemma 5 we know that for every eigenvalue λ of A/α, there exists $1 \leqslant t \leqslant l$ such that

$$|\lambda - a'_{tt}| \leqslant P_t^{\gamma}(A/\alpha)S_t^{1-\gamma}(A/\alpha). \tag{3.4}$$

Hence

$$\begin{aligned}
0 \geqslant{}& |\lambda - a'_{tt}| - P_t^{\gamma}(A/\alpha)S_t^{(1-\gamma)}(A/\alpha) \\
={}& |\lambda - a'_{tt}| - \left(\sum_{\substack{s=1\\ s\neq t}}^{l} |a'_{ts}|\right)^{\gamma}\left(\sum_{\substack{s=1\\ s\neq t}}^{l} |a'_{st}|\right)^{1-\gamma} \\
={}& \left|\lambda - a_{j_tj_t} + (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\
&- \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_tj_s} - (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_s}\\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\gamma} \\
&\cdot \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_sj_t} - (a_{j_si_1}, \ldots, a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\right]^{(1-\gamma)}
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\geqslant{}& |\lambda - a_{j_tj_t}| - \left|(a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right| \\
&- \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_tj_s} - (a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_s}\\ \vdots \\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\gamma} \\
&\cdot \left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_sj_t} - (a_{j_si_1}, \ldots, a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|\right]^{(1-\gamma)}.
\end{aligned} \tag{3.6}$$

From the proof of Theorem 2 we know

$$\left|(a_{j_ti_1}, \ldots, a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix} a_{i_1j_t}\\ \vdots \\ a_{i_kj_t}\end{pmatrix}\right|$$

$$
\begin{aligned}
&+\left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_tj_s}-(a_{j_ti_1},\ldots,a_{j_ti_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_s}\\ \vdots\\ a_{i_kj_s}\end{pmatrix}\right|\right]^{\gamma}\\
&\cdot\left[\sum_{\substack{s=1\\ s\neq t}}^{l}\left|a_{j_sj_t}-(a_{j_si_1},\ldots,a_{j_si_k})[A(\alpha)]^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|\right]^{(1-\gamma)}\\
&\leqslant (P_{j_t}(A)-w_{j_t})^{\gamma}(S_{j_t}(A)-w_{j_t}^{T})^{1-\gamma}.
\end{aligned}
$$

Therefore, from (3.5),

$$
\begin{aligned}
0 &\geqslant |\lambda-a'_{tt}|-P_t^{\gamma}(A/\alpha)S_t^{1-\gamma}(A/\alpha)\\
&\geqslant |\lambda-a_{j_tj_t}|-(P_{j_t}(A)-w_{j_t})^{\gamma}(S_{j_t}(A)-w_{j_t}^{T})^{1-\gamma}.
\end{aligned}
$$

Thus we obtain (3.3).

From Theorem 4 we obtain the following corollary. □

Corollary 3. *Let $A\in\mathbb{C}^{n\times n}$, $N_r(A)\cap N_c(A)\neq\emptyset$, $\alpha=\{i_1,i_2,\ldots,i_k\}\subseteq N_r(A)\cap N_c(A)$, $\alpha'=N-\alpha=\{j_1,j_2,\ldots,j_l\}$ and $k=|\alpha|<n, l=n-k$. Then for any $0\leqslant\gamma\leqslant 1$ and every eigenvalue λ of A/α, there exists $1\leqslant t\leqslant l$ such that*

$$
\begin{aligned}
|\lambda-a_{j_tj_t}| &\leqslant \gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-\gamma w_{j_t}-(1-\gamma)w_{j_t}^{T}\\
&\leqslant \gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A).
\end{aligned}
$$

Proof. Using the mean value inequality, from Theorem 4, there exists $1\leqslant t\leqslant l$ such that

$$
\begin{aligned}
|\lambda-a_{j_tj_t}| &\leqslant (P_{j_t}(A)-w_{j_t})^{\gamma}(S_{j_t}(A)-w_{j_t}^{T})^{1-\gamma}\\
&\leqslant \gamma(P_{j_t}(A)-w_{j_t})+(1-\gamma)(S_{j_t}(A)-w_{j_t}^{T})\\
&= \gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A)-\gamma w_{j_t}-(1-\gamma)w_{j_t}^{T}\\
&\leqslant \gamma P_{j_t}(A)+(1-\gamma)S_{j_t}(A). \qquad \square
\end{aligned}
$$

4. Numerical example

In this section, we first give an example to estimate the bounds for eigenvalues of the Schur complement with the entries of the original matrix. Then we give another example to reveal the advantages of the Schur-based iteration.

Example 2. Let

$$
A=\begin{pmatrix}15&2&3&4&5\\ 2&20&8&4&3\\ 3&4&5&7&2\\ 4&5&1&2&6\\ 5&1&2&3&4\end{pmatrix},\quad \alpha=\{1,2\}.
$$

If we estimate the bounds for eigenvalues of A/α by the entries of A/α, there would be a large amount of computations to do. However, as

$$
\begin{aligned}
&P_1(A)=14;\quad P_2(A)=17;\quad P_3(A)=16;\quad P_4(A)=16;\quad P_5(A)=11;\\
&S_1(A)=14;\quad S_2(A)=12;\quad S_3(A)=14;\quad S_4(A)=18;\quad S_5(A)=16;
\end{aligned}
$$

$$w_3 = \min\left\{\frac{|a_{11}| - P_1(A)}{|a_{11}|}, \ \frac{|a_{22}| - P_2(A)}{|a_{22}|}\right\}\sum_{i=1}^{2}|a_{3i}| = \frac{7}{15};$$

$$w_4 = \min\left\{\frac{|a_{11}| - P_1(A)}{|a_{11}|}, \ \frac{|a_{22}| - P_2(A)}{|a_{22}|}\right\}\sum_{i=1}^{2}|a_{4i}| = \frac{3}{5};$$

$$w_5 = \min\left\{\frac{|a_{11}| - P_1(A)}{|a_{11}|}, \ \frac{|a_{22}| - P_2(A)}{|a_{22}|}\right\}\sum_{i=1}^{2}|a_{5i}| = \frac{2}{5};$$

$$w_3^T = \min\left\{\frac{|a_{11}| - S_1(A)}{|a_{11}|}, \ \frac{|a_{22}| - S_2(A)}{|a_{22}|}\right\}\sum_{i=1}^{2}|a_{i3}| = \frac{11}{15};$$

$$w_4^T = \min\left\{\frac{|a_{11}| - S_1(A)}{|a_{11}|}, \ \frac{|a_{22}| - S_2(A)}{|a_{22}|}\right\}\sum_{i=1}^{2}|a_{i4}| = \frac{8}{15};$$

$$w_5^T = \min\left\{\frac{|a_{11}| - S_1(A)}{|a_{11}|}, \ \frac{|a_{22}| - S_2(A)}{|a_{22}|}\right\}\sum_{i=1}^{2}|a_{i5}| = \frac{8}{15}.$$

Since $\alpha \in N_r(A)$, according to Theorem 3, the eigenvalue z of A/α satisfies

$$z \in \{z||z-5| \leqslant 15.53\} \cup \{z||z-2| \leqslant 15.4\} \cup \{z||z-4| \leqslant 10.6\} \equiv G_1.$$

On the other hand, since $\alpha \in N_r(A) \cap N_c(A)$, if we take $\gamma = \frac{1}{2}$, by Theorem 4 the eigenvalue z of A/α satisfies

$$z \in \{z||z-5| \leqslant 14.36\} \cup \{z||z-2| \leqslant 16.40\} \cup \{z||z-4| \leqslant 12.80\} \equiv G_2.$$

Denote Rez the real part of z. Then $-13.4 \leqslant Rez \leqslant 20.53$ for $z \in G_1$ and $-14.4 \leqslant Rez \leqslant 19.36$ for $z \in G_2$. So we see that $G_1 \not\subset G_2$ and $G_2 \not\subset G_1$.

Example 3. Consider a system of linear equation $Mx = b$, where

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad b = \begin{pmatrix} 3 & 3 & \dots & 3 \end{pmatrix}^T_{1\times 100},$$

$$A = \begin{pmatrix} 120 & -60 & & & \\ -60 & 120 & -60 & & \\ & \ddots & \ddots & \ddots & \\ & & -60 & 120 & -60 \\ & & & -60 & 120 \end{pmatrix}_{50\times 50},$$

$$B = C^T = \begin{pmatrix} 0 & 0 & \dots & 0 & 60 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \dots & 0 & 0 \\ -60 & 0 & \dots & 0 & 0 \end{pmatrix}_{50\times 50},$$

$$D = \begin{pmatrix} 51\times 120 & -600 & & & & \\ -600 & 52\times 120 & -600 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -600 & 98\times 120 & -600 & \\ & & & -600 & 99\times 120 & -600 \\ & & & & -600 & 1200000 \end{pmatrix}_{50\times 50}.$$

Table 1
Results of example 3($\varepsilon = 10^{-6}$). Computer condition: Pentium(R) 4 CPU 3.2GHz, extended memory 512 M.

	CGM	SCGM
x_1	1.338752	1.250313
x_2	2.623017	2.450567
x_3	3.848661	3.600819
x_4	5.011911	4.701071
x_5	6.109618	5.751322
x_6	7.139409	6.751573
x_7	8.099808	7.701822
x_8	8.990306	8.602070
x_9	9.811391	9.452317
x_{10}	10.564522	10.252563
x_{11}	11.252054	11.002809
x_{12}	11.877123	11.703053
x_{13}	12.443488	12.353298
x_{14}	12.955343	12.953541
x_{15}	13.417109	13.503785
x_{16}	13.833208	14.004029
x_{17}	14.207848	14.454274
x_{18}	14.544822	14.854519
x_{19}	14.847129	15.204765
x_{20}	15.118822	15.505012
x_{21}	15.348622	15.755260
x_{22}	15.605010	15.955510
x_{23}	15.689931	16.105753
x_{24}	15.705309	16.204590
x_{25}	15.707172	16.255644
x_{26}	15.707189	16.257428
x_{27}	15.705470	16.208498
x_{28}	15.690838	16.107315
x_{29}	15.607772	15.957555
x_{30}	15.350868	15.757803
x_{31}	15.121572	15.508053
x_{32}	14.850169	15.208303
x_{33}	14.548211	14.858554
x_{34}	14.211614	14.458806
x_{35}	13.837391	14.009059
x_{36}	13.421750	13.509312
x_{37}	12.960488	12.959566
x_{38}	12.449180	12.359819
x_{39}	11.883403	11.710073
x_{40}	11.258960	11.010325
x_{41}	10.572084	10.260577
x_{42}	9.819632	9.460829
x_{43}	8.999238	8.611079
x_{44}	8.109436	7.711328
x_{45}	7.149727	6.761576
x_{46}	6.120610	5.761823
x_{47}	5.023552	4.712069
x_{48}	3.860920	3.612315
x_{49}	2.635856	2.462559
x_{50}	1.352129	1.262803
x_{51}	0.013772	0.013047

Table 1
(*continued*)

	CGM	SCGM
x_{52}	0.001773	0.001802
x_{53}	0.000752	0.000696
x_{54}	0.000739	0.000579
x_{55}	0.000720	0.000557
x_{56}	0.000649	0.000544
x_{57}	0.000577	0.000532
x_{58}	0.000529	0.000521
x_{59}	0.000504	0.000510
x_{60}	0.000492	0.000500
x_{61}	0.000485	0.000490
x_{62}	0.000478	0.000481
x_{63}	0.000471	0.000472
x_{64}	0.000462	0.000463
x_{65}	0.000454	0.000455
x_{66}	0.000446	0.000446
x_{67}	0.000439	0.000439
x_{68}	0.000431	0.000431
x_{69}	0.000424	0.000424
x_{70}	0.000417	0.000417
x_{71}	0.000410	0.000410
x_{72}	0.000403	0.000403
x_{73}	0.000397	0.000397
x_{74}	0.000391	0.000391
x_{75}	0.000385	0.000385
x_{76}	0.000379	0.000379
x_{77}	0.000373	0.000373
x_{78}	0.000368	0.000368
x_{79}	0.000362	0.000362
x_{80}	0.000357	0.000357
x_{81}	0.000352	0.000352
x_{82}	0.000347	0.000347
x_{83}	0.000342	0.000342
x_{84}	0.000339	0.000338
x_{85}	0.000331	0.000333
x_{86}	0.00033	0.000329
x_{87}	0.000319	0.000325
x_{88}	0.000327	0.000321
x_{89}	0.000312	0.000316
x_{90}	0.000309	0.000313
x_{91}	0.000322	0.000309
x_{92}	0.000286	0.000305
x_{93}	0.000310	0.000301
x_{94}	0.000310	0.000298
x_{95}	0.000274	0.000294
x_{96}	0.000290	0.000291
x_{97}	0.000310	0.000287
x_{98}	0.000279	0.000283
x_{99}	0.000240	0.000264
x_{100}	−0.000064	−0.000060
Cputime	0.03125 s	0.015625 s
$\|Mx - b\|_2$	14.510824	0.232572

Obviously, M is real symmetric and irreducibly diagonally dominant with positive diagonal entries. So M is positive definite (see, [16, p. 23]). Thus we can solve this system with the conjugate gradient method.

Furthermore, by Theorem 1.12 of [17], D and M/D are also positive definite. Consequently, we can convert the original system into the following systems by using the Schur complement

$$M/Dy = f, \tag{4.1}$$
$$Dz = g - Cy, \tag{4.2}$$

where

$$y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{50} \end{pmatrix}, \quad f = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{50} \end{pmatrix} - BD^{-1} \begin{pmatrix} b_{51} \\ b_{52} \\ \vdots \\ b_{100} \end{pmatrix}, \quad z = \begin{pmatrix} x_{51} \\ x_{52} \\ \vdots \\ x_{100} \end{pmatrix}, \quad g = \begin{pmatrix} b_{51} \\ b_{52} \\ \vdots \\ b_{100} \end{pmatrix}.$$

Then we can first solve (4.1) and then (4.2) by the conjugate gradient method. We call this method the Schur-based conjugate gradient method.

As M/D, D and M are all nonsingular, the rank of M is greater than that of M/D and D. On the other hand, we know from Theorem 3 that the eigenvalues of M/D and D are more concentrated than those of M. So we predict that the Schur-based conjugate gradient method will compute faster than the ordinary conjugate gradient method (see, e.g., [18, pp. 312–317]).

In fact, solving the original system by the conjugate gradient method needs 90 iteration steps and it takes 0.031250 seconds' cputime to compute out x; solving (4.1) and (4.2) by the conjugate gradient method needs 26 and 16 iteration steps, respectively and it takes 0.015625 seconds' total cputime to compute out x.

The results of computation are given out in Table 1, from which we see that the Schur-based conjugate gradient method (SCGM) is much better than the ordinary conjugate gradient method (CGM).

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